Dual Control Theory

DCS Control Theory Reading Seminar

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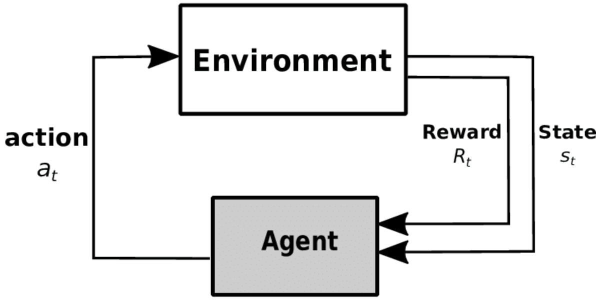
Motivation

When travelling to a new place, how do you choose what to eat?

Multi-armed bandit: Which slot machine gives me more money

Administering medicine dosage to patient



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How much should I invest and how quickly I should act?

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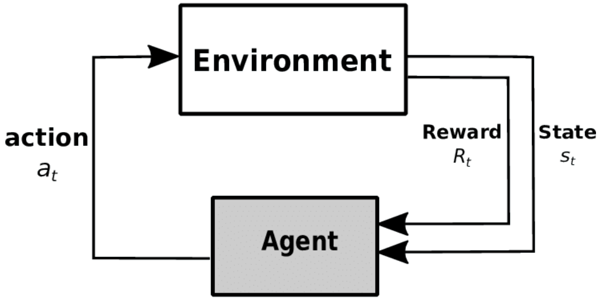

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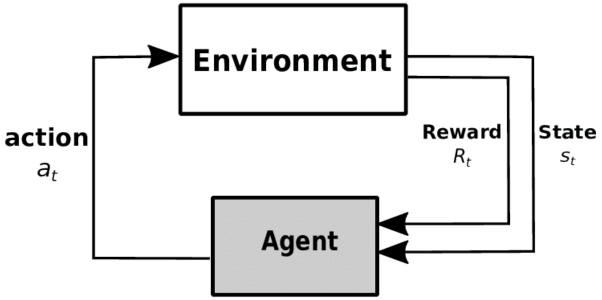
Motivation

When travelling to a new place, how do you choose what to eat?

How much should I invest and how quickly I should act?

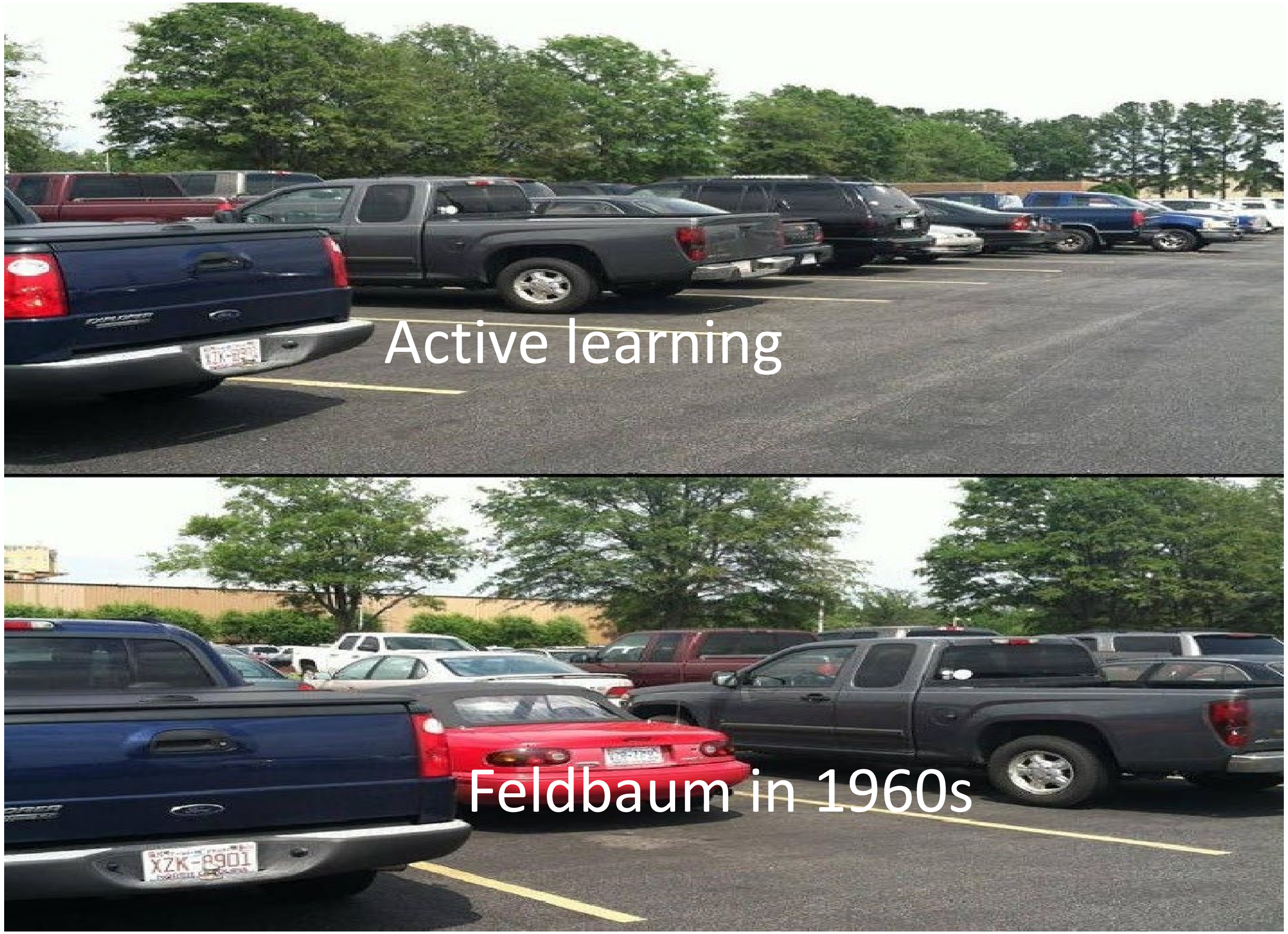
(Active learning) Dual control

Multi-armed bandit: Which slot machine gives me more money



Administering medicine dosage to patient

Why is it relevant now?



Brief Bio – Alexander Fel’dbaum

Birth and Education:

- Born on August 16, 1913, in Yekaterinoslav (now Ukraine).
- Graduated from Moscow Power Engineering Institute in 1937.
- Defended his PhD thesis on the theory of controlling devices in 1943.
- Worked in Peter the Great Military Academy of the Strategic Missile Forces after 1945.
- Defended his doctoral dissertation on the dynamics of automatic regulation systems in 1953.
- Passed away in 1969, in Moscow.

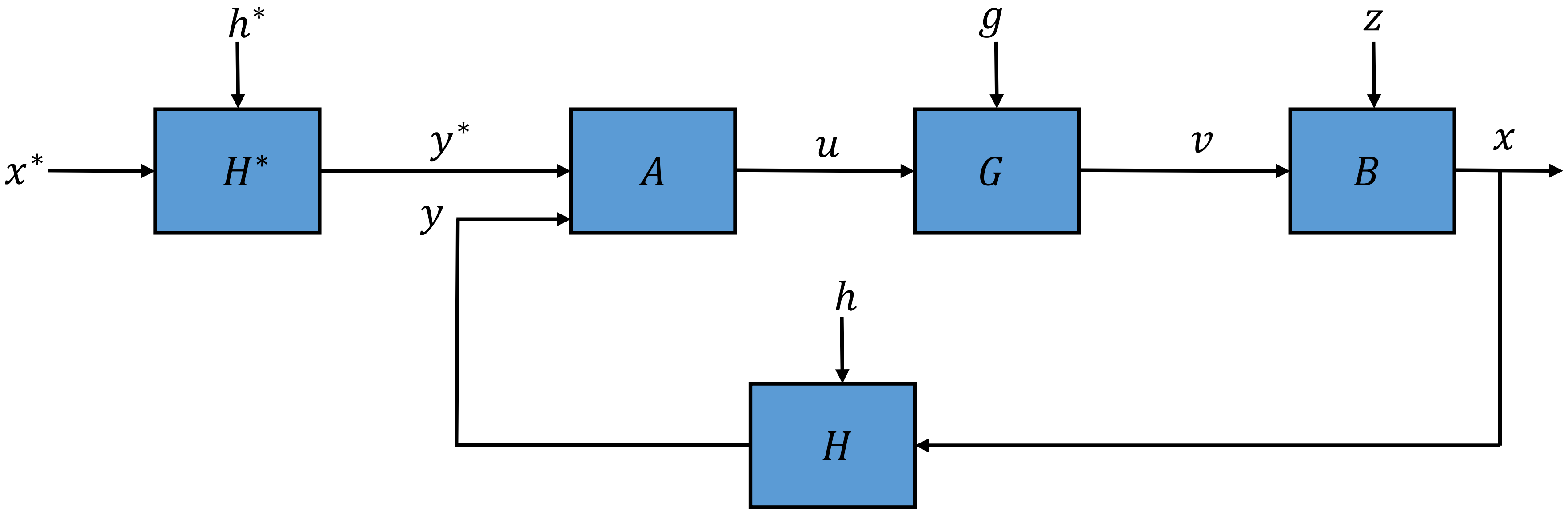
Contributions:

- Dual control theory:
- Addressing exploration-exploitation trade-off in systems with unknown characteristics.
- Foundational in reinforcement learning and adaptive control methods.

Outline

- ❑ Problem setup
- ❑ Open loop case
  - ➢ Derivation of risk
  - ➢ Derivation of optimum strategy
- ❑ Closed loop case
  - ➢ Derivation of risk
  - ➢ Derivation of optimum strategy
- ❑ Conclusion

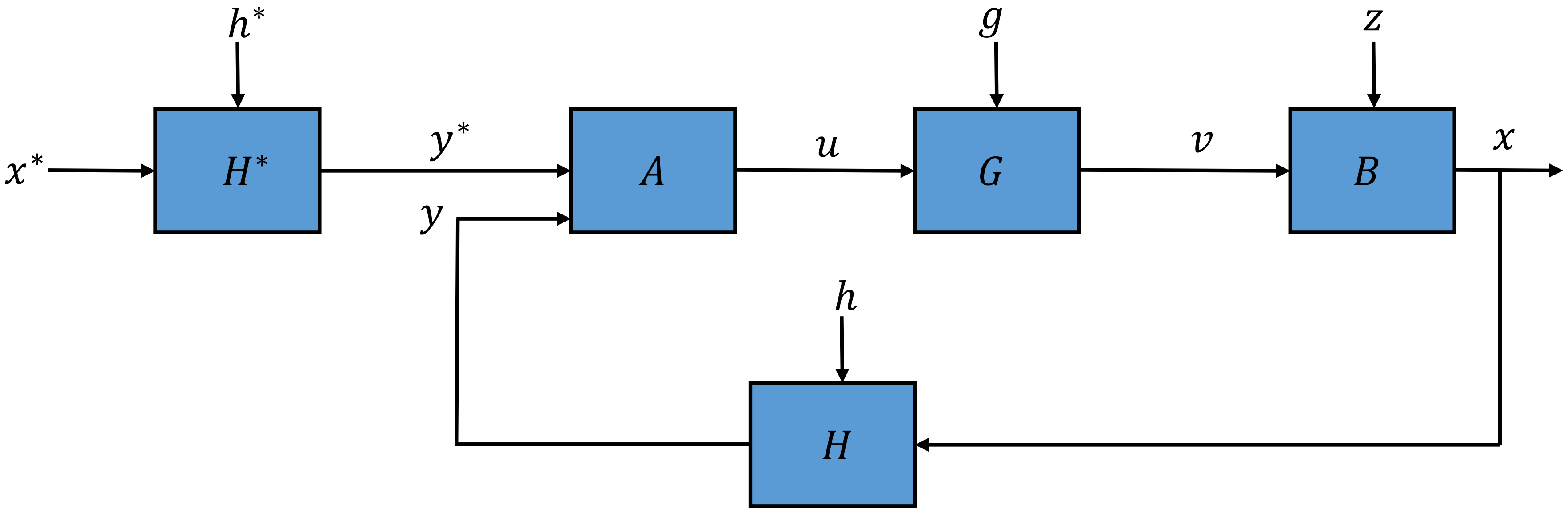
Problem setup



Design the control process (u_s) such that it is

- Investigational: Obtain the information on the characteristics of B
- Directional: drive B to a desired state

Problem setup



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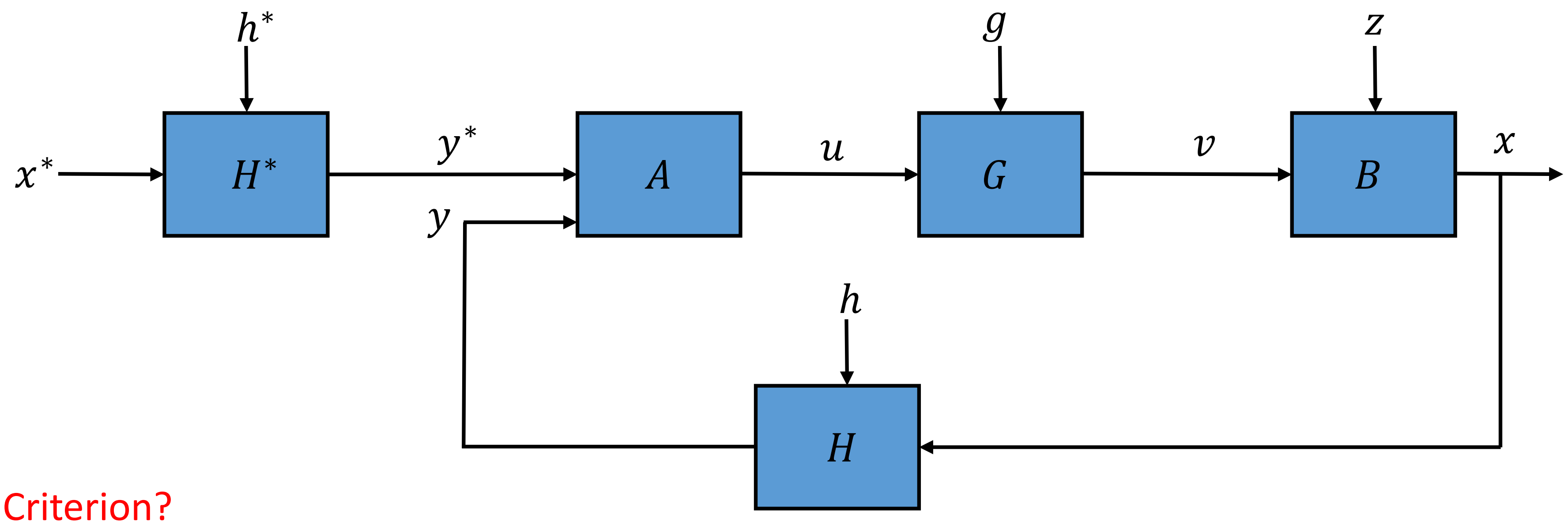
$\Bigg\}$ Exploration-exploitation trade off

Problem setup



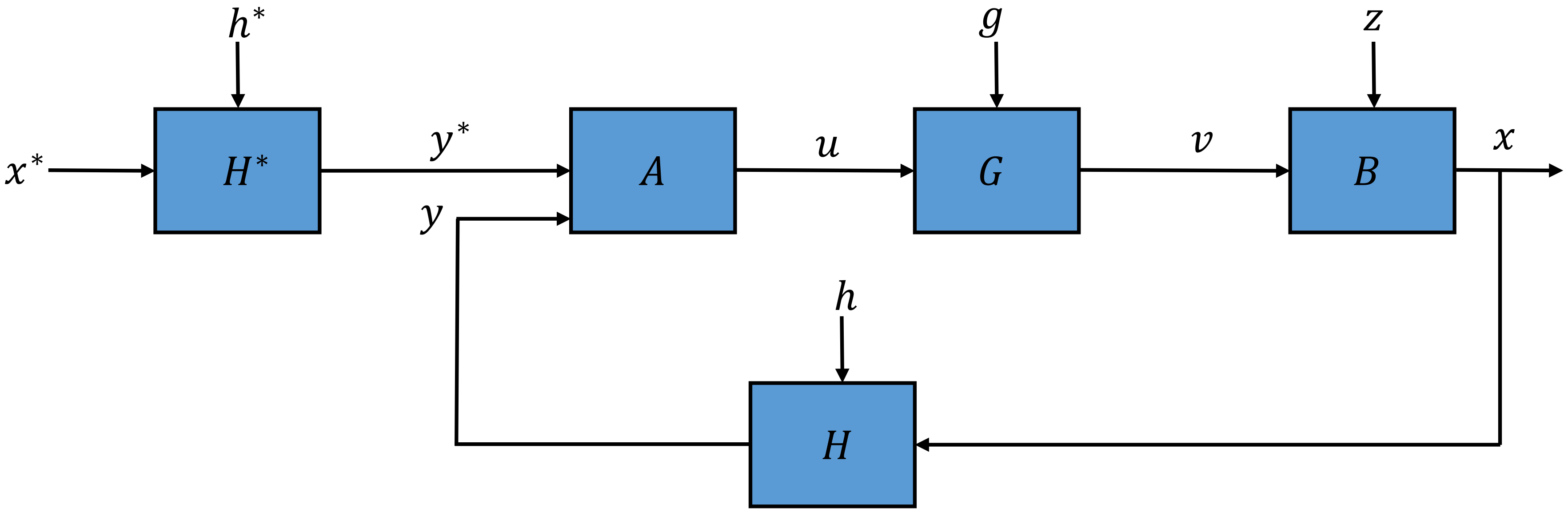
Criterion?

Design the control process (u_s) such that it is

- Investigational: Obtain the information on the characteristics of B
- Directional: drive B to a desired state

Exploration-exploitation trade off

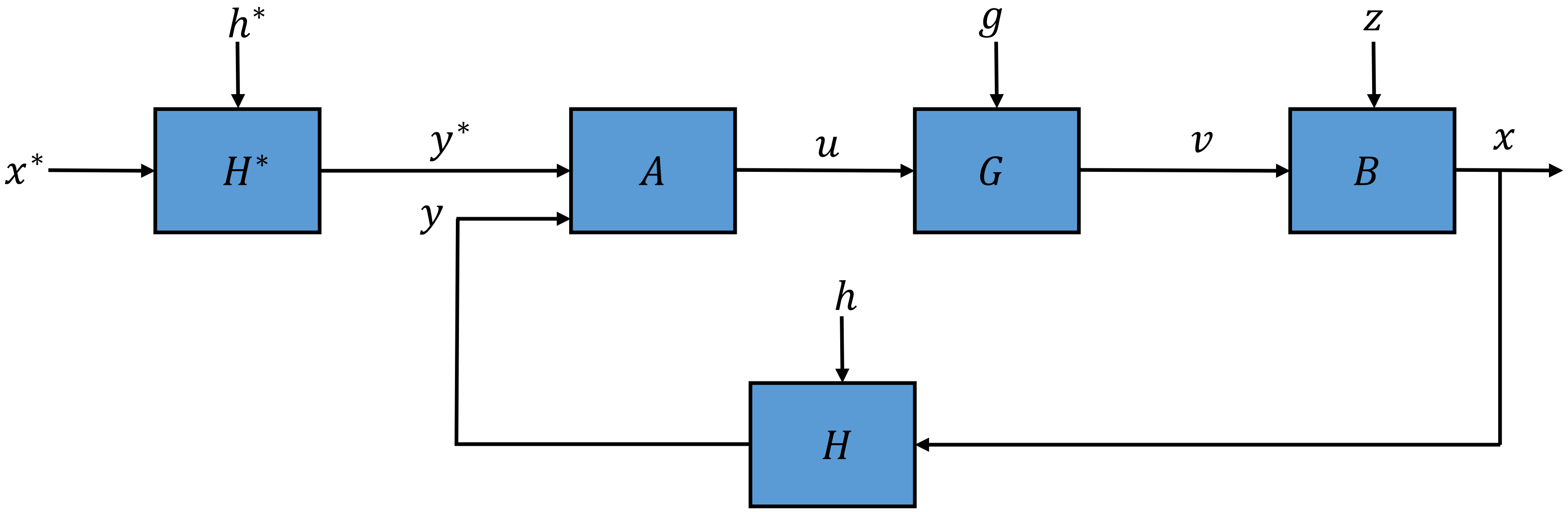
Problem setup



Goal: Find a sequence of probability densities $\Gamma_s(u_s|\boldsymbol{u}_{s-1}, \boldsymbol{y}_s^*, \boldsymbol{y}_{s-1})$ that minimizes

$$R = \mathbb{E}(W) = \sum_{s=0}^{s=n} \mathbb{E}(W_s) = \sum_{s=0}^{s=n} R_s$$

Problem setup



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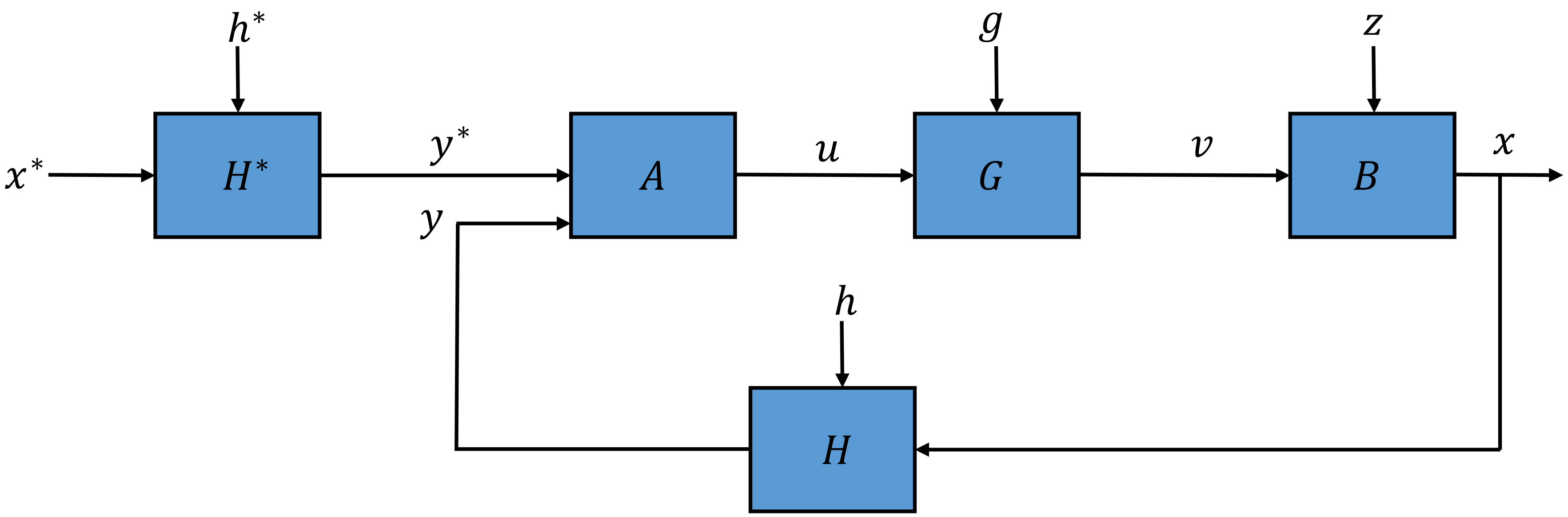
Risk

Problem setup



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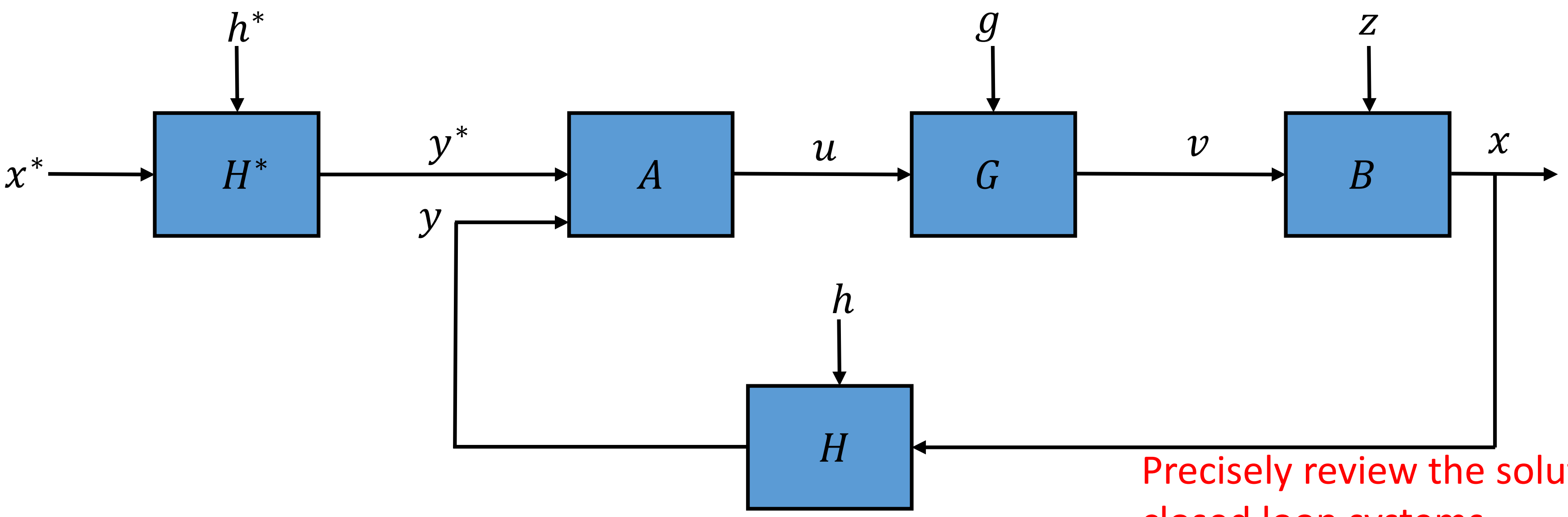
$$R = \mathbb{E}(W) = \sum_{s=0}^{n} \mathbb{E}(W_s) = \sum_{s=0}^{n} R_s$$

Total loss $W = \sum_{s=0}^{n} W_s$, W_s - partial loss, say for eg. $W_s = \alpha(s)\,(x_s - x_s^*)^2$

Problem setup



Precisely review the solution for open and closed loop systems

Goal: Find a sequence of probability densities $\Gamma_s(u_s|\boldsymbol{u}_{s-1}, \boldsymbol{y}_s^*, \boldsymbol{y}_{s-1})$ that minimizes

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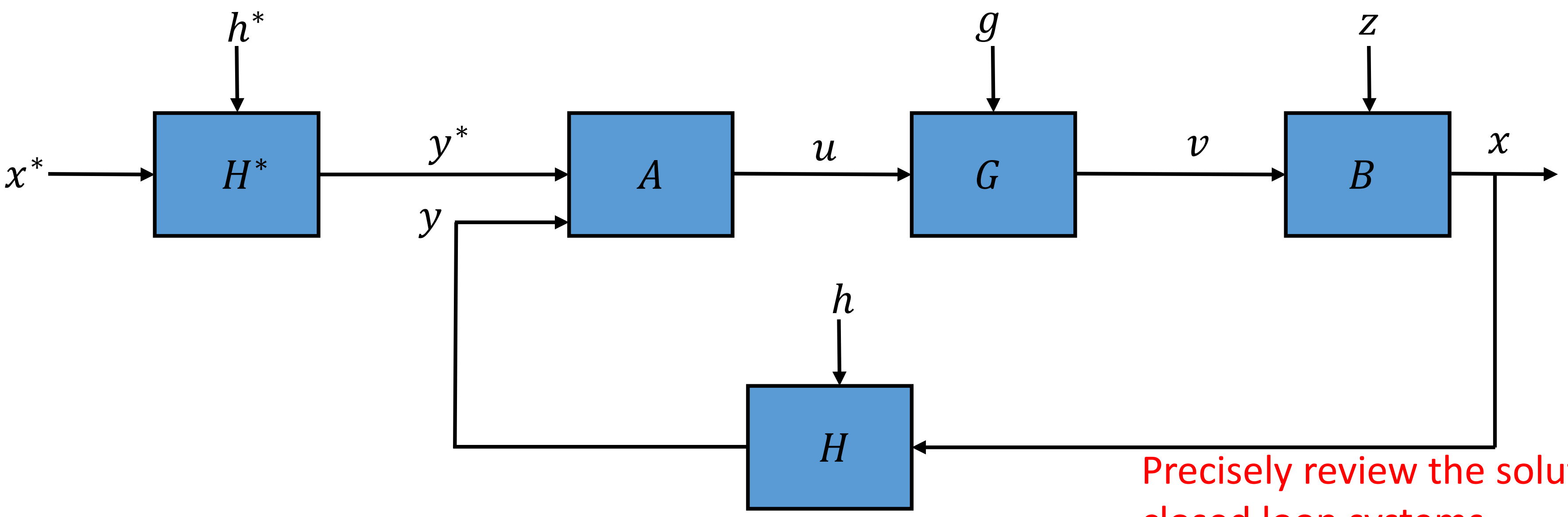
Problem setup



Mathematical tools

- Bayes rule
- Dynamic programming

Precisely review the solution for open and closed loop systems

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Outline

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- Closed loop case
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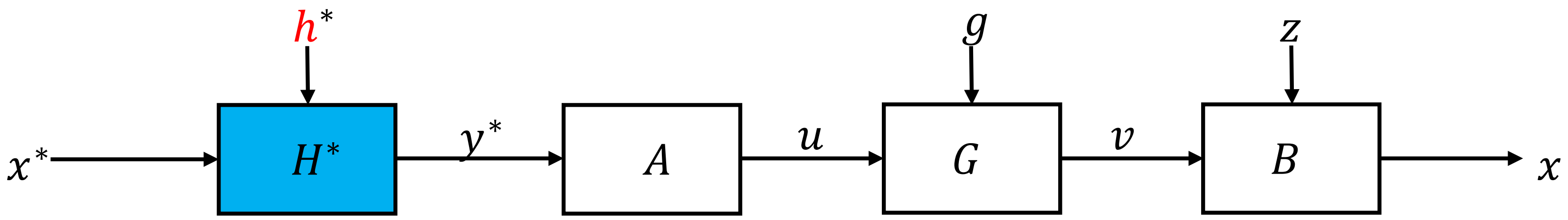
- Conclusion

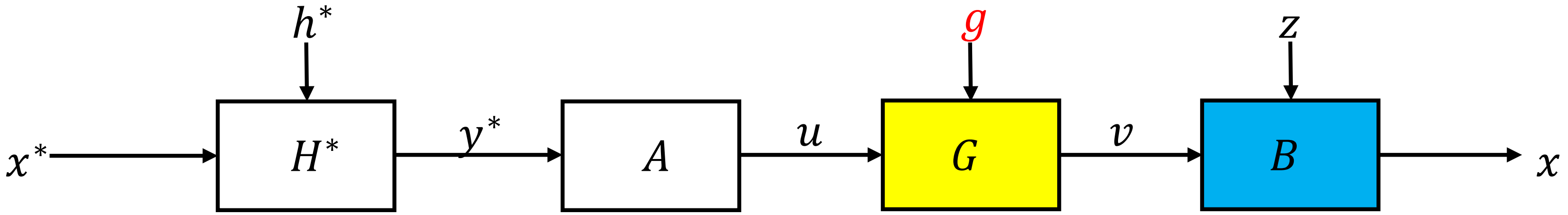
Derivation of risk



- The input: $x_s^* = x_s^*(s, \lambda\ )$
- The time moment: s = 0,1, $\cdots$, n; where n is fixed
- The parameter vector $\lambda = (\lambda_1, \cdots, \lambda_q)$
- The noise h^*

- Priori information:
- $P_0(\lambda) = P(\lambda)$,
- h^*'s statistical properties, $P(h_s^*)$ is assume to be fixed for s^*,
- The way of combing x^* and h^*,
- $P(y_s^*|x_s^*)$, being identical for all s.

Derivation of risk



The controlled object B

$x_s = F_0(z_s, v_s)$, F_0 is known,

where $z_s = z_s(s, \mu)$,

The parameter vector $\mu = (\mu_1, \cdots, \mu_m)$ with $P_0(\mu) = P(\mu)$,

The noise g

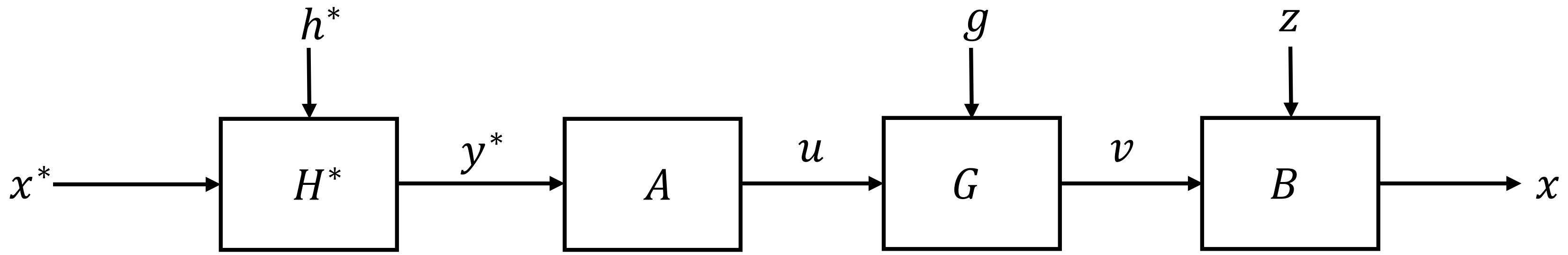
Priori information:

g_s's statistical properties, $P(g_s)$ is assume to be fixed for s^*,

The way of combing u_s and g_s

$P(v_s^* | u_s^*)$

Derivation of risk



Goal: Find a sequence of probability densities $\Gamma_s(u_s|y^*_{s-1})$ that the average risk

$$R = \mathbb{E}(W) = \sum_{s=0}^{s=n} \mathbb{E}(W_s) = \sum_{s=0}^{s=n} R_s\,;$$

is minimum.

Derivation of risk

The conditional partial risk:

$$r_s = \mathbb{E}(W_s|x_s^*)$$

$$= \int_{\Omega(x_s,v_s,u_s,y_{s-1}^*)} W_s(s, x_s^*, x_s)P(x_s|v_s)P(v_s|u_s)\,\Gamma_s(u_s|y_{s-1}^*) \cdot P(y_{s-1}^*|x_{s-1}^*)d\Omega(x_s, v_s, u_s, y_{s-1}^*)$$

where

$P(y_{s-1}^*|x_{s-1}^*)$=$\prod_{i=0}^{i=s-1} P(y_i^*|x_i^*)$

Derivation of risk

The partial risk $R_s = \int_{\Omega(\lambda)} r_s \; P(\lambda) d\Omega(\lambda)$

$$= \int_{\Omega(x_s, v_s, u_s, y^*_{s-1})} P(x_s|v_s) P(v_s|u_s)\, \Gamma_s(u_s|y^*_{s-1}) \cdot \rho_s(x_s, y^*_{s-1}) d\Omega(x_s, v_s, u_s, y^*_{s-1})$$

where

$\rho_s(x_s, y^*_{s-1})$ =$\int_{\Omega(\lambda)} W_s(s, x^*_s(s,\lambda), x_s) P(y^*_{s-1}|x^*_{s-1}) P(\lambda)\, d\lambda$

The total risk

$$R = \sum R_s = \sum_{s=0}^{s=n} \int_{\Omega(x_s, v_s, u_s, y^*_{s-1})} P(x_s|v_s) P(v_s|u_s)\, \Gamma_s(u_s|y^*_{s-1}) \cdot \rho_s(x_s, y^*_{s-1}) d\Omega(x_s, v_s, u_s, y^*_{s-1})$$

Determination of optimum strategy

The total risk

$$R = \sum R_s = \sum_{s=0}^{s=n} \int_{\Omega(x_s, v_s, u_s, y_{s-1}^*)} P(x_s|v_s)P(v_s|u_s)\,\Gamma_s(u_s|y_{s-1}^*)$$
$$\cdot\, \rho_s(x_s, y_{s-1}^*)\,d\Omega(x_s, v_s, u_s, y_{s-1}^*)$$

- Function $\Gamma_s(u_s|y_{s-1}^*)$ only affect R_s for fixed s
- Select Γ_s such that R_s is minimum

Determination of optimum strategy

The total risk

$$R_s = \int_{\Omega(x_s, v_s, u_s, y^*_{s-1})} P(x_s|v_s)P(v_s|u_s)\,\Gamma_s(u_s|y^*_{s-1})$$
$$\cdot\, \rho_s(x_s, y^*_{s-1})d\Omega(x_s, v_s, u_s, y^*_{s-1})$$

$$R_s = \int_{\Omega(u_s)} I(y^*_{s-1})d\Omega(y^*_{s-1}),$$

where

$$I(y^*_{s-1}) = \int_{\Omega(u_s)} \Gamma_s(u_s|y^*_{s-1})\xi_s(u_s, y^*_{s-1})d\Omega(u_s)$$

with

$$\xi_s(u_s, y^*_{s-1}) = \int_{\Omega(x_s s, v_s)} P(x_s|v_s)P(v_s|u_s)\rho(x_s, y^*_{s-1})d\Omega(x_s, v_s)$$

Determination of optimum strategy

$$I(y_{s-1}^{*}) = \int_{\Omega(u_s)} \Gamma_s(u_s|y_{s-1}^{*})\xi_s(u_s, y_{s-1}^{*})d\Omega(u_s) = \mathbb{E}(\xi_s) \geq (\xi_s)_{\min}$$

Then u_s^* is selected when $\xi_s(u_s^*, y_{s-1}^*)$= $(\xi_s)_{\min}$.

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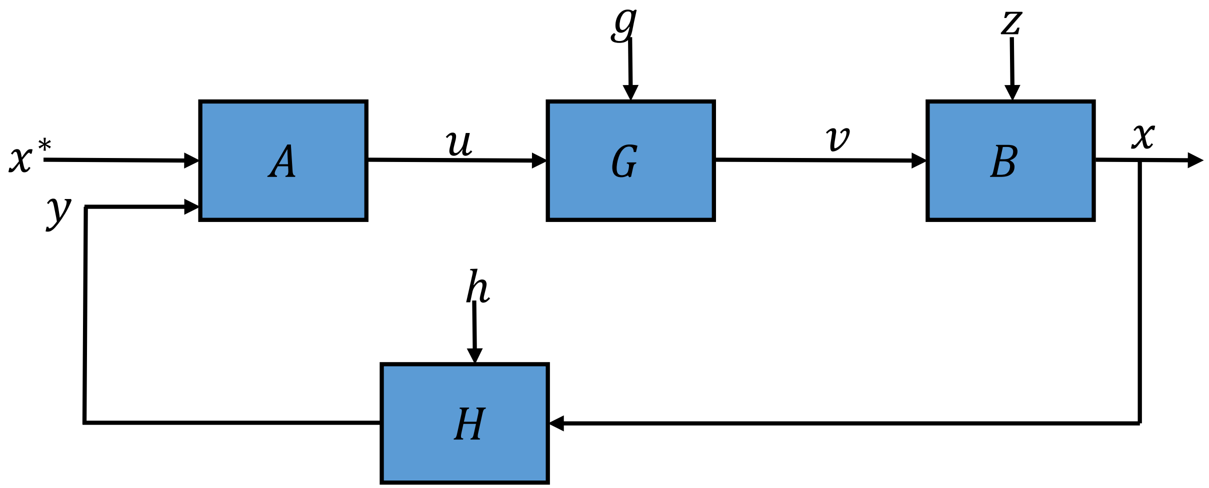
Derivation of the risk



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is minimum.

Derivation of risk

The conditional partial risk:

$$
\begin{aligned}
r_s &= \mathbb{E}[W_s | x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1}] \\
&= \int_{\Omega(\mathrm{x_s}, \mathrm{v}_s, u_s)} W_s(s, x_s^*, x_s)\, P_s(x_s|v_s) P(v_s|u_s) \Gamma_s(u_s|x_s^*, y_{s-1}, u_{s-1}\,) d\Omega(x_s, v_s, u_s)
\end{aligned}
$$

The partial risk:

$$
\begin{aligned}
R_s &= \mathbb{E}[r_s] \\
&= \int_{\Omega(\mathrm{x_s}, \mathrm{v}_s, \boldsymbol{u}_s, \mathrm{y}_{s-1})} W_s(s, x_s^*, x_s)\, P_s(x_s|v_s) P(v_s|u_s) \Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1}\,) \\
&\qquad \cdot P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\, d\Omega(x_s, v_s\,, \boldsymbol{u}_s, \boldsymbol{y}_{s-1})
\end{aligned}
$$

Derivation of risk

Let's look at the partial risk more carefully

$$R_s \;= \int_{\Omega(\mathrm{x_s},\mathrm{v}_s,\boldsymbol{u}_s,\mathrm{y}_{s-1})} W_s(s, x_s^*, x_s)\, P_s(x_s|v_s) P(v_s|u_s) \Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1}\,)$$

$$\cdot\, P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\, d\Omega(x_s, v_s\,, \boldsymbol{u}_s, \boldsymbol{y}_{s-1})$$

Derivation of risk

Let's look at the partial risk more carefully

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Derivation of risk

Let's look at the partial risk more carefully

$$R_s = \int_{\Omega(\mathrm{x}_s,\boldsymbol{u}_s,\mathrm{y}_{s-1})} W_s(s, x_s^*, x_s) \left[\int_{\Omega(v_s)} P_s(x_s|v_s) P(v_s|u_s)\ d\Omega(v_s)\right] \Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1}) \cdot P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\, d\Omega(x_s, \boldsymbol{u}_s, \boldsymbol{y}_{s-1})$$

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$$\int_{\Omega(v_s)} P_s(x_s|v_s) P(v_s|u_s)\ d\Omega(v_s) = \int_{\Omega(\mu)} P(x_s|\boldsymbol{\mu}, u_s)\ P_s(\mu)\, d\Omega(\mu)$$

Derivation of risk

Let's look at the partial risk more carefully

$$R_s = \int_{\Omega(\mathrm{x_s},\boldsymbol{u}_s,\mathrm{y}_{s-1})} W_s(s, x_s^*, x_s)\left[\int_{\Omega(v_s)} P_s(x_s|v_s)P(v_s|u_s)\ d\Omega(v_s)\right]\Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1})$$
$$\cdot P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{u}_s, \boldsymbol{y}_{s-1})$$

$$\int_{\Omega(v_s)} P_s(x_s|v_s)P(v_s|u_s)\ d\Omega(v_s) = \int_{\Omega(\mu)} P(x_s|\mu, u_s)\ P_s(\mu)\ d\Omega(\mu)$$

Now the partial risk becomes

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s)P_s(\boldsymbol{\mu})\Gamma_{\mathrm{s}}(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

Derivation of risk

Let's look at the partial risk more carefully

$$R_s = \int_{\Omega(\mathrm{x}_s, \boldsymbol{u}_s, \mathrm{y}_{s-1})} W_s(s, x_s^*, x_s) \left[\int_{\Omega(v_s)} P_s(x_s|v_s) P(v_s|u_s)\ d\Omega(v_s) \right] \Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1}) \cdot P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\, d\Omega(x_s, \boldsymbol{u}_s, \boldsymbol{y}_{s-1})$$

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Now the partial risk becomes

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Posterior update of $\boldsymbol{\mu}$

Derivation of risk

$$R_s = \int_{\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\; P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u}_s|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\; P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\; d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

Prior

Bayes rule: $P_s(\mu) = P(\boldsymbol{\mu}|\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}) = \dfrac{P(\boldsymbol{\mu}) P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\boldsymbol{\mu})}{P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})}$

Derivation of risk

$$R_s = \int_{\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u}_s|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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$$P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\mu) = P(u_{s-1}, y_{s-1}|\mu, u_{s-2}, y_{s-2}) P(u_{s-2}, y_{s-2}|\mu, u_{s-3}, y_{s-3}) \ldots P(u_i, y_i|\mu, u_{i-1}, y_{i-1}) \ldots P(u_0, y_0|\mu)$$

Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s)P_s(\boldsymbol{\mu})\Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

Bayes rule: $P_s(\mu) = P(\boldsymbol{\mu}|\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}) = \dfrac{P(\boldsymbol{\mu})P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\boldsymbol{\mu})}{P(\boldsymbol{u}_{s-1},\boldsymbol{y}_{s-1})}$

$$P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\mu) = P(u_{s-1}, y_{s-1}|\mu, u_{s-2}, y_{s-2})P(u_{s-2}, y_{s-2}|\mu, u_{s-3}, y_{s-3})$$
$$\ldots P(u_i, y_i|\mu, u_{i-1}, y_{i-1}) \ldots P(u_0, y_0|\mu)$$

Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s)P_s(\boldsymbol{\mu})\Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

Bayes rule: $P_s(\mu) = P(\boldsymbol{\mu}|\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}) = \frac{P(\boldsymbol{\mu})P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\boldsymbol{\mu})}{P(\boldsymbol{u}_{s-1},\boldsymbol{y}_{s-1})}$

$$P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1}|\mu) = P(u_{s-1}, y_{s-1}|\mu, u_{s-2}, y_{s-2})P(u_{s-2}, y_{s-2}|\mu, u_{s-3}, y_{s-3}) \ldots P(u_i, y_i|\mu, u_{i-1}, y_{i-1}) \ldots P(u_0, y_0|\mu)$$

$$P(u_i, y_i|\mu, u_{i-1}, y_{i-1}) = P(y_i|\mu, u_i, u_{i-1}, y_{i-1})P(u_i|\mu, u_{i-1}, y_{i-1})$$

Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\; P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_{\mathrm{s}}(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\; P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\; d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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$$\begin{aligned} P(u_i, y_i|\mu, u_{i-1}, y_{i-1}) &= P(y_i|\mu, u_i, u_{i-1}, y_{i-1}) P(u_i|\mu, u_{i-1}, y_{i-1}) \\ &= P(y_i|\mu, i, u_i) P(u_i|\mu, u_{i-1}, y_{i-1}) \end{aligned}$$

Derivation of risk

$$R_s = \int_{\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s,x_s^*,x_s)\; P(x_s|\boldsymbol{\mu},u_s)P_s(\boldsymbol{\mu})\Gamma_s(\mathrm{u}_s|\boldsymbol{x}_{s-1}^*,\mathbf{u}_{s-1},\mathbf{y}_{s-1})\; P(\boldsymbol{u}_{s-1},\boldsymbol{y}_{s-1})\; d\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})$$

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Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s,x_s^*,x_s)\ P(x_s|\boldsymbol{\mu},u_s)P_s(\boldsymbol{\mu})\Gamma_\mathrm{s}(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*,\mathbf{u}_{s-1},\mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1},\boldsymbol{y}_{s-1})\ d\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})$$

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$$P(\boldsymbol{u}_{s-1},\boldsymbol{y}_{s-1}|\mu) = \left[\prod_{i=0}^{s-1} P(y_i|\mu,i,u_i)\right]\left[\prod_{i=1}^{s-1}\Gamma_i\ (u_i|u_{i-1},y_{i-1})\right]P_0(u_0)$$

Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s) P_s(\boldsymbol{\mu}) \Gamma_s(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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$\coloneqq \Gamma_0$ (given)

Derivation of risk

$$R_s = \int_{\Omega(x_s,\boldsymbol{\mu},u_s,\boldsymbol{y}_{s-1})} W_s(s, x_s^*, x_s)\ P(x_s|\boldsymbol{\mu}, u_s)P_s(\boldsymbol{\mu})\Gamma_\mathrm{s}(\mathrm{u_s}|\boldsymbol{x}_{s-1}^*, \mathbf{u}_{s-1}, \mathbf{y}_{s-1})\ P(\boldsymbol{u}_{s-1}, \boldsymbol{y}_{s-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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$\coloneqq \Gamma_0$ (given)

$$R_s = \int_{\Omega(x_s,\mu,u_s,y_{s-1})} W_s(s, x_s^*, x_s)\, P(x_s|\mu, u_s)P(\mu) \prod_{i=0}^{s-1} P(y_i|\mu, i, u_i) \prod_{i=1}^{s} \Gamma_i\ (u_i|u_{i-1}, y_{i-1})\ d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

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Total risk is then obtained as

$$R = \sum_{s=0}^{n} \int_{\Omega(x_s,\mu,u_s,y_{s-1})} W_s(s, x_s, x_s^*)\;\, P(x_s|\mu, u_s)P(\mu) \prod_{i=0}^{s-1} P(y_i|\mu, i, u_i) \prod_{i=1}^{s} \Gamma_i\,(u_i|u_{i-1}, y_{i-1})\;\, d\Omega(x_s, \boldsymbol{\mu}, u_s, \boldsymbol{y}_{s-1})$$

- Γ_s influences the term R_i, for $i > s$
- $\sum_{i=s+1}^{n} R_i$ represents the investigation risk
  - Γ_s causes either a worse or better investigation of the characteristics of B
- In open loop, Γ_s influences just R_s - Risk associated here is just action/directional

Determination of Optimum Strategy

Use of Dynamic programming to find the optimal sequence of probability densities $\Gamma_s(u_s | x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1})$.

We start at k=n and then do backward iteration k=n-1,…

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Recall

$$R_n = \int_{\Omega(x_n,\boldsymbol{\mu},\boldsymbol{u}_n,\boldsymbol{y}_{n-1})} W_n(n, x_n^*, x_n)\ P(x_n|\boldsymbol{\mu}, u_n)P(\boldsymbol{\mu}) \prod_{i=0}^{n-1} P(y_i|\boldsymbol{\mu}, i, u_i) \prod_{i=0}^{n-1} \Gamma_i\ \Gamma_n\, d\Omega(x_n, \boldsymbol{\mu}, \boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

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Idea: Assume $\Gamma_0, \dots, \Gamma_{n-1}$ are given, find Γ_n such that $S_n := R_n$ is minimum and it satisfies $\int_{\Omega(u_n)} \Gamma_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) d\Omega(u_n) = 1$

Determination of Optimum Strategy

Use of Dynamic programming to find the optimal sequence of probability densities $\Gamma_s(u_s|x_s^*, \boldsymbol{y}_{s-1}, \boldsymbol{u}_{s-1})$.

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$$R_n = \int_{\Omega(x_n, \boldsymbol{\mu}, \boldsymbol{u}_n, \boldsymbol{y}_{n-1})} W_n(n, x_n^*, x_n)\ P(x_n|\boldsymbol{\mu}, u_n) P(\boldsymbol{\mu}) \prod_{i=0}^{n-1} P(y_i|\boldsymbol{\mu}, i, u_i) \prod_{i=0}^{n-1} \Gamma_i\ \Gamma_n\ d\Omega(x_n, \boldsymbol{\mu}, \boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

$u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}$

Idea: Assume $\Gamma_0, \dots, \Gamma_{n-1}$ are given, find Γ_n such that $S_n := R_n$ is minimum and it satisfies $\int_{\Omega(u_n)} \Gamma_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) d\Omega(u_n) = 1$

Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

where

$$\alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) = \int_{\Omega(x_n, \boldsymbol{\mu},)} W_n(n, x_n^*, x_n)\ P(x_n|\boldsymbol{\mu}, u_n) P(\boldsymbol{\mu}) \prod_{i=0}^{n-1} P(y_i|\boldsymbol{\mu}, i, u_i)\ d\Omega(x_n, \boldsymbol{\mu})$$

$$\beta_k = \prod_{i=0}^{k} \Gamma_i$$

Further, $R_n = \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \beta_{n-1} \kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$

where $\kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) = \int_{\Omega(u_n)} \alpha_n(u_n, u_{n-1}, y_{n-1}) \Gamma_n(u_n, u_{n-1}, y_{n-1})\ d\Omega(u_n)$

Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n(u_n,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n,\boldsymbol{y}_{n-1})$$

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minimize

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Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

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Determination of Optimum Strategy

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minimize known minimize

Further, $R_n = \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \beta_{n-1} \kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \; d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$

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Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n(u_n,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n,\boldsymbol{y}_{n-1})$$

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Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

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Probability density

Determination of Optimum Strategy

$$R_n = \int_{\Omega(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ \beta_{n-1}\ \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

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$$\beta_k = \prod_{i=0}^{k} \Gamma_i$$

Optimal solution

$u_n^* = \arg \min_{u_n \in \Omega(u_n)} \alpha_n(u_n, u_{n-1}, y_{n-1})$

minimize known minimize

Further, $R_n = \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \beta_{n-1} \kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\ d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$

where $\boxed{\kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) = \int_{\Omega(u_n)} \alpha_n(u_n, u_{n-1}, y_{n-1}) \Gamma_n(u_n, u_{n-1}, y_{n-1})\ d\Omega(u_n)}$

Probability density

Determination of Optimum Strategy

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Optimal solution

$$u_n^* = \arg \min_{u_n \in \Omega(u_n)} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$$

Further, $R_n = \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \beta_{n-1} \kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \ d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$

where $\kappa_n(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) = \int_{\Omega(u_n)} \alpha_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \Gamma_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \ d\Omega(u_n)$

$$\boxed{\Gamma_n(u_n, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) = \delta(u_n - u_n^*)}$$

Determination of Optimum Strategy

$k = n - 2$

$$\int_{\Omega(u_n^*, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n \, \beta_{n-1} \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

$$\int_{\Omega(u_{n-1}, \boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2})} \alpha_{n-1} \, \beta_{n-2} \Gamma_{n-1} d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-2})$$

Determination of Optimum Strategy

$k = n - 2$

$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$

$$\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n \beta_{n-1} \Gamma_n d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

$$S_{n-1} := R_{n-1} + R_n$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\, \beta_{n-2} \Gamma_{n-1} d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-2})$$

Determination of Optimum Strategy

$k = n - 2$

$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$

$$\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n\,\beta_{n-1}\,\Gamma_n\,d\Omega(\boldsymbol{u}_n,\boldsymbol{y}_{n-1})$$

$\Gamma_n = \delta(u_n - u_n^*)$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\,\beta_{n-2}\Gamma_{n-1}\,d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-2})$$

Determination of Optimum Strategy

$k = n - 2$

$$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$$

$$\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n\,\beta_{n-1}\,\Gamma_n\, d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})$$

$$S_{n-1} \coloneqq R_{n-1} + R_n \qquad \Gamma_n = \delta(u_n - u_n^*)$$

$$= \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\,\alpha_n(u_n^*, u_{n-1}, y_{n-1})\; d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\,\beta_{n-2}\Gamma_{n-1}\, d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-2})$$

Determination of Optimum Strategy

$k = n - 2$

$$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$$

$$\boxed{\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n\, \beta_{n-1}\, \Gamma_n\, d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})}$$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

$$\Gamma_n = \delta(u_n - u_n^*)$$

$$\boxed{= \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1})\; d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})}$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\, \beta_{n-2}\Gamma_{n-1} d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-2})$$

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \left\{ \begin{array}{c} \displaystyle\int_{\Omega(u_{n-1})} \Gamma_{n-1}\alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \displaystyle\int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1}) d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \end{array} \right\}$$

Determination of Optimum Strategy

$k = n - 2$

$$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$$

$$\boxed{\int_{\Omega(u_n^*, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \alpha_n\, \beta_{n-1} \Gamma_n\, d\Omega(\boldsymbol{u}_n, \boldsymbol{y}_{n-1})}$$

$$\Gamma_n = \delta(u_n - u_n^*)$$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

minimize

$$\boxed{= \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1})\; d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})}$$

$$\int_{\Omega(u_{n-1}, \boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2})} \alpha_{n-1}\, \beta_{n-2} \Gamma_{n-1}\, d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-2})$$

$$= \int_{\Omega(\boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2})} \beta_{n-2} \left\{ \begin{array}{c} \int_{\Omega(u_{n-1})} \Gamma_{n-1} \alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \int_{\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1}) d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \end{array} \right\}$$

Determination of Optimum Strategy

$k = n - 2$

$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$

$$\boxed{\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n\,\beta_{n-1}\,\Gamma_n\, d\Omega(\boldsymbol{u}_n,\boldsymbol{y}_{n-1})}$$

$\Gamma_n = \delta(u_n - u_n^*)$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

minimize

$$\boxed{= \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\,\alpha_n(u_n^*, u_{n-1}, y_{n-1})\; d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})}$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\,\beta_{n-2}\Gamma_{n-1} d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-2})$$

fixed

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \left\{ \begin{array}{c} \displaystyle\int_{\Omega(u_{n-1})} \Gamma_{n-1}\alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \displaystyle\int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\,\alpha_n(u_n^*, u_{n-1}, y_{n-1}) d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1}) \end{array} \right\}$$

Determination of Optimum Strategy

$k = n - 2$

$$\beta_{n-1} = \beta_{n-2}\,\Gamma_{n-1}$$

$$\boxed{\int_{\Omega(u_n^*,\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \alpha_n\,\beta_{n-1}\,\Gamma_n\, d\Omega(\boldsymbol{u}_n,\boldsymbol{y}_{n-1})}$$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

minimize

$$\Gamma_n = \delta(u_n - u_n^*)$$

$$\boxed{= \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\,\alpha_n(u_n^*,u_{n-1},y_{n-1})\ d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})}$$

$$\int_{\Omega(u_{n-1},\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \alpha_{n-1}\,\beta_{n-2}\Gamma_{n-1}\, d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-2})$$

fixed

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \underbrace{\left\{ \begin{array}{c} \int_{\Omega(u_{n-1})} \Gamma_{n-1}\alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\,\alpha_n(u_n^*,u_{n-1},y_{n-1})d\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1}) \end{array} \right\}}_{\text{minimize}}$$

Determination of Optimum Strategy

$k = n - 2$

$$S_{n-1} := R_{n-1} + R_n$$

minimize

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \left\{ \begin{array}{c} \int_{\Omega(u_{n-1})} \Gamma_{n-1}\alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1}) d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \end{array} \right\}$$

fixed

probability density

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \boxed{\Gamma_{n-1}(u_{n-1}, \boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2})}$$

$$\cdot \underbrace{\left\{ \alpha_{n-1} + \int_{\Omega(\boldsymbol{y}_{n-1})} \alpha(u_n^*, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\, d\Omega(\boldsymbol{y}_{n-1}) \right\}}_{:= \gamma_{n-1}} d\Omega(\boldsymbol{u}_{n-1})$$

minimize

Determination of Optimum Strategy

$k = n - 2$

$$S_{n-1} \coloneqq R_{n-1} + R_n$$

minimize

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2} \left\{ \begin{array}{c} \int_{\Omega(u_{n-1})} \Gamma_{n-1}\alpha_{n-1}\, d\Omega(u_{n-1}) + \\ \int_{\Omega(\boldsymbol{u}_{n-1},\boldsymbol{y}_{n-1})} \Gamma_{n-1}\, \alpha_n(u_n^*, u_{n-1}, y_{n-1}) d\Omega(\boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1}) \end{array} \right\}$$

fixed

probability density

$$= \int_{\Omega(\boldsymbol{u}_{n-2},\boldsymbol{y}_{n-2})} \beta_{n-2}\, \boxed{\Gamma_{n-1}(u_{n-1}, \boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2})}$$

$$\cdot \underbrace{\left\{ \alpha_{n-1} + \int_{\Omega(\boldsymbol{y}_{n-1})} \alpha(u_n^*, \boldsymbol{u}_{n-1}, \boldsymbol{y}_{n-1})\, d\Omega(\boldsymbol{y}_{n-1}) \right\}}_{\coloneqq\, v_{n-1}} d\Omega(\boldsymbol{u}_{n-1})$$

minimize

Optimal solution:

$$u_{n-1}^* = \arg \min_{u_{n-1} \in \Omega(u_{n-1})} v_{n-1}$$

$$\Gamma_{n-1}(u_{n-1}, \boldsymbol{u}_{n-2}, \boldsymbol{y}_{n-2}) = \delta(u_{n-1} - u_{n-1}^*)$$

Determination of Optimum Strategy

$k = n - i$

$$(S_{n-i})_{\min} := \left(\sum_{j=0}^{n-i} R_{n-j} \right)_{\min} = \int_{\Omega(u_{n-i-1}, y_{n-i-1})} \beta_{n-i-1}\, v_{n-i}^{*}\, d\Omega(\boldsymbol{u}_{n-i-1}, \boldsymbol{y}_{n-i-1})$$

$$v_{n-i} = \alpha_{n-i} + \int_{\Omega(\boldsymbol{y}_{n-i})} v_{n-i+1}\,(u_{n-i+1}^{*}, \boldsymbol{u}_{n-i}, \boldsymbol{y}_{n-i}) d\Omega(\boldsymbol{y}_{n-i})$$

Optimal solution:
$u_{n-i}^{*} = \arg\min_{u_{n-i} \in \Omega(u_{n-i})} v_{n-i}(u_{n-i}; \boldsymbol{u}_{n-i-1}, \boldsymbol{y}_{n-i-1})$

$\Gamma_{n-i} = \delta(u_{n-i} - u_{n-i}^{*})$

Determination of Optimum Strategy

$k = n - i$

$$(S_{n-i})_{\min} := \left(\sum_{j=0}^{n-i} R_{n-j}\right)_{\min} = \int_{\Omega(u_{n-i-1}, y_{n-i-1})} \beta_{n-i-1}\, v_{n-i}^{*}\, d\Omega(\boldsymbol{u}_{n-i-1}, \boldsymbol{y}_{n-i-1})$$

$$v_{n-i} = \alpha_{n-i} + \int_{\Omega(\boldsymbol{y}_{n-i})} v_{n-i+1}\,(u_{n-i+1}^{*}, \boldsymbol{u}_{n-i}, \boldsymbol{y}_{n-i}) d\Omega(\boldsymbol{y}_{n-i})$$

Again, the optimum strategy proves to be not random but a regular one

Optimal solution:
$u_{n-i}^{*} = \arg\min_{u_{n-i} \in \Omega(u_{n-i})} v_{n-i}(u_{n-i}; \boldsymbol{u}_{n-i-1}, \boldsymbol{y}_{n-i-1})$

$\Gamma_{n-i} = \delta(u_{n-i} - u_{n-i}^{*})$

Outline

- Problem setup
- Open loop case
  - Derivation of risk
  - Derivation of optimum strategy
- Closed loop case
  - Derivation of risk
  - Derivation of optimum strategy
- Conclusion

Conclusion

- We reviewed Feldbaum's dual control theory (part I and part II) seminal paper
- How exploration-exploitation dilemma was addressed via careful formulation of risk, followed by the optimum strategy
- In open loop case, we saw that the optimum strategy is regular, directional
- In closed loop case, the optimum strategy is still regular but both investigational and directional

THANK YOU FOR YOUR ATTENTION ☺