Linear Estimation in Krein Spaces

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Motivation: H^{∞} Problem

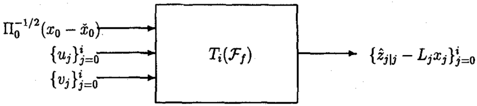
$$\begin{cases} x_{i+1} = F_i x_i + G_i u_i, & x_0 \\ y_i = H_i x_i + v_i, & i \geq 0 \end{cases}$$

- Goal: To estimate some arbitrary linear combination of the states, say

$$z_i = L_i x_i$$

- A Posteriori estimate: $\check{z}_{i|i} = \mathcal{F}_f\left(y_0, y_1, \cdots, y_i\right)$
- How do we gauge the "quality" of the above estimate?

Motivation: H^∞ Problem



H^∞ Problem

Find estimation strategies $\check{z}_{i|i} = \mathcal{F}_f\left(y_0, y_1, \cdots, y_i\right)$ that achieve $\left\|T_i\left(\mathcal{F}_f\right)\right\|_\infty < \gamma_f \; (\gamma_f > 0)$

$$\left\|T_i\left(\mathcal{F}_f\right)\right\|_\infty = \sup_{x_0, u \in h_2, v \in h_2} \frac{\sum_{j=0}^{i} e_{f,j}^* e_{f,j}}{\left(x_0 - \check{x}_0\right)^* \Pi_0^{-1}\left(x_0 - \check{x}_0\right) + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j}$$

[2] "Linear estimation in Krein spaces. II. Applications" Hassibi, B.; Sayed, A.H.; Kailath, T. Automatic Control, IEEE Transactions on Volume: 41 1, Jan. 1996, Page(s): 34 -49

Motivation: H^∞ Problem

$$\|T\|_\infty := \sup_{u \in h_2, u \neq 0} \frac{\|Tu\|_2}{\|u\|_2},$$

where $\|u\|_2$ is the h_2-norm of the causal sequence $\{u_k\}$, i.e., $\|u\|_2^2 = \sum_{k=0}^{\infty} u_k^* u_k$

- Interpretation: Maximum energy gain from the input u to the output y
- Idea: Filter $\mathcal{F}_f$ to H^∞ can be obtained as certain Kalman filter via suitable construction of state-space model
- Caution: Projections cannot be realized in Hilbert space unlike standard Kalman filter.

Remedy?

Projection in Krein Space!

- Projections in Krein Space often lead to indefinite quadratic forms $\Longrightarrow$ need additional conditions to ensure uniqueness and minimum.

Outline

Outline

Krein Space

Projections and Quadratic Forms

Back to H^{∞}: Krein Space Solution

Conclusion

Krein Space

An abstract vector space $\{\mathcal{K}, \langle\cdot,\cdot\rangle\}$ that satisfies the following requirements is called a Krein Space:
i) $\mathcal{K}$ is a linear space over $\mathcal{C}$, the complex numbers.
ii) $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle^*$,
iii) $\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle = a\langle \boldsymbol{x}, \boldsymbol{z} \rangle + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle$
for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{K}, a, b \in \mathcal{C}$
iv) The vector space $\mathcal{K}$ admits a direct orthogonal sum decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$$

such that $\{\mathcal{K}_+, \langle\cdot,\cdot\rangle\}$ and $\{\mathcal{K}_-, -\langle\cdot,\cdot\rangle\}$ are Hilbert spaces, and

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$$

for any $x \in \mathcal{K}_+$and $y \in \mathcal{K}_-$.
Note: If $\boldsymbol{x}, \boldsymbol{y}$ are stochastic variables, then $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \mathbb{E}[\boldsymbol{x}\boldsymbol{y}^*]$

[1] "Linear estimation in Krein spaces. I. Theory" Hassibi, B.; Sayed, A.H.; Kailath, T. Automatic Control, IEEE Transactions on Volume: 41 1, Jan. 1996, Page(s): 18 -33

Krein Space

Krein space looks like Hilbert space except that

- $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \not\Rightarrow \boldsymbol{x} = 0$ ($\boldsymbol{x}$ is called Neutral vector)
- There exists $\boldsymbol{x} \neq \boldsymbol{0}$ in a linear subspace (M) of $\mathcal{K}$ such that $\boldsymbol{x} \perp M$ ($\boldsymbol{x}$ is called Isotropic vector)
- $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ can be negative

Outline

Projections in Krein Spaces

Definition

Given $\boldsymbol{z}$, $\{\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_N\}$ in $\mathcal{K}$, $\hat{\boldsymbol{z}}$ is projection of $\boldsymbol{z}$ onto $\mathcal{L}\{\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_N\}$ if

$$\boldsymbol{z} = \hat{\boldsymbol{z}} + \tilde{\boldsymbol{z}}$$

where $\hat{\boldsymbol{z}} \in \mathcal{L}\{\boldsymbol{y}_0, \cdots, \boldsymbol{y}_N\}$ and $\tilde{\boldsymbol{z}}$ satisfies the orthogonality condition

$$\tilde{\boldsymbol{z}} \perp \mathcal{L}\{\boldsymbol{y}_0, \cdots, \boldsymbol{y}_N\}$$

or equivalently, $\langle \tilde{\boldsymbol{z}}, \boldsymbol{y}_i \rangle = 0$ for $i = 0, 1, \cdots, N$.

Remark:

- Projections always exist and unique in Hilbert space
- In Krein space, existence and uniqueness of projection require additional conditions

Projections in Krein Spaces

Lemma

If the Gramian matrix $R_y = \langle \boldsymbol{y}, \boldsymbol{y} \rangle$ is nonsingular, then the projection of $\boldsymbol{z}$ onto $\mathcal{L}\{\boldsymbol{y}\}$ exists, is unique, and is given by

$$\hat{\boldsymbol{z}} = \langle \boldsymbol{z}, \boldsymbol{y} \rangle \langle \boldsymbol{y}, \boldsymbol{y} \rangle^{-1} \boldsymbol{y} = R_{zy} R_y^{-1} \boldsymbol{y}.$$

So, standing assumption: R_y is nonsingular

Projections and Quadratic Forms

- In Hilbert space, projections minimize certain quadratic forms
- In Krein space, projections stationarize certain quadratic forms
- Need additional condition for stationary point to be minimum

To this end, we look at two closely related problems,

- Stochastic minimization problem
- A partially equivalent deterministic problem

Stochastic Minimization Problem

A natural quadratic form to study: error Gramian

$$P(k) = \langle \mathbf{z} - \mathbf{k}^* \mathbf{y}, \mathbf{z} - k^* \mathbf{y} \rangle \,,$$

i) $\{\mathbf{y}_0, \cdots, \mathbf{y}_N\}$: collection of elements in a Krein space $\mathcal{K}$ with indefinite inner product $\langle \cdot, \cdot \rangle$,
ii) $\mathbf{z} = \text{col}\,\{z_0, \cdots, z_M\}$: some column vector of elements in $\mathcal{K}$,
iii) $k^* \mathbf{y}$ an arbitrary linear combination of $\{\mathbf{y}_0, \cdots, \mathbf{y}_N\}$. $\mathbf{y} = \text{col}\,\{\mathbf{y}_0, \cdots, \mathbf{y}_N\}$

$$P(k) = \|\mathbf{z} - \hat{\mathbf{z}}\|^2_{\mathcal{K}} + \|\hat{\mathbf{z}} - \mathbf{k}^* \mathbf{y}\|^2_{\mathcal{K}}$$

where $\hat{\mathbf{z}} = k_0^* \mathbf{y} = R_{zy} R_y^{-1}$ is the projection of $\mathbf{z}$ onto $\mathcal{L}(\mathbf{y})$

Stochastic Minimization Problem

- $\left\|\hat{\boldsymbol{z}} - k^*\boldsymbol{y}\right\|^2 = \left\|k_0^*\boldsymbol{y} - k^*\boldsymbol{y}\right\|^2 = 0$, even if $k_0 \neq k$.
- $\left\|k_0^*\boldsymbol{y} - k^*\boldsymbol{y}\right\|^2$ could be negative, $P(k)$ may not be minimized by choosing $k = k_0$

Theorem 1

When R_y is nonsingular, k_0, the unique coefficient matrix in the projection of z onto $\mathcal{L}\{\boldsymbol{y}\}$, $\hat{\boldsymbol{z}} = k_0^*\boldsymbol{y},\ k_0 = R_y^{-1}R_{yz}$ yields the unique stationary point of the error Gramian

$$P(k) \triangleq \langle \boldsymbol{z} - k^*\boldsymbol{y}, \boldsymbol{z} - k^*\boldsymbol{y} \rangle$$
$$= \begin{bmatrix} I & -k^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} \begin{bmatrix} I \\ -k \end{bmatrix}$$

Further, k_0 is a unique minimum iff $R_y > 0$ i.e., R_y is not only nonsingular but also positive definite.

A Partially Equivalent Deterministic Problem

- Scalar second-order form:

$$J(z,y) \triangleq \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix}$$

Theorem 2

Suppose both R_y and the block matrix in above equation are nonsingular. Then
a) The stationary point z_0 of $J(z,y)$ over z is given by

$$z_0 = R_{zy} R_y^{-1} y.$$

b) The value of $J(z,y)$ at the stationary point is

$$J\left(z_0, y\right) = y^* R_y^{-1} y$$

Further, z_0 is a minimum iff $R_z - R_{zy} R_y^{-1} R_{yz} > 0$.

Implications of Theorem 1 and 2

- Stationary point for the scalar second-order form is “same" as that in Theorem 1 for Krein space projection of vector $\boldsymbol{z}$ onto $\mathcal{L}(\boldsymbol{y})$
- H^{∞} problems will lead directly to certain indefinite quadratic forms:
  - Theorem 1 provides algorithm to find stationary point via corresponding Krein-space projection
  - Theorem 2 is used to check if stationary point obtained above is indeed the minimum

Outline

Back to H^{∞}: Krein Space Solution

- Recall H^{∞} problem: $\|T_i(\mathcal{F}_f)\|_{\infty} < \gamma_f$
- For all nonzero $\left\{x_0, \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i}\right\}$

  $$\frac{\sum_{j=0}^{i} \left|\check{z}_{j|j} - L_j x_j\right|^2}{(x_0 - \check{x}_0)^* \Pi_0^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |y_j - H_j x_j|^2} < \gamma_f^2$$

  $\implies$ for all $k \leq i$

  $$\frac{\sum_{j=0}^{k} \left|\check{z}_{j|j} - L_j x_j\right|^2}{(x_0 - \check{x}_0)^* \Pi_0^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{k} |u_j|^2 + \sum_{j=0}^{k} |y_k - H_i x_i|^2} < \gamma_f^2.$$

- The above inequality is an indefinite quadratic form

Back to H^∞: Krein Space Solution

Lemma 3

$\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$ iff there exists $\check{z}_{k|k} = \mathcal{F}_f(y_0, \cdots, y_k)$ (for all $0 \leq k \leq i$) such that for all complex vectors x_0, for all causal sequences $\{u_j\}_{j=0}^{i}$, and for all nonzero causal sequences $\{y_j\}_{j=0}^{i}$, the scalar quadratic form

$$
\begin{aligned}
& J_{f,k}(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k) \\
&= x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k} u_j^* u_j \\
&\quad + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j) \\
&\quad - \gamma_f^{-2} \sum_{j=0}^{k} (\check{z}_{j|j} - L_j x_j)^* (\check{z}_{j|j} - L_j x_j)
\end{aligned}
$$

satisfies $J_{f,k}(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k) > 0$ for all $0 \leq k \leq i$

Back to H^{∞}: Krein Space Solution

To show that $J_{f,k}\left(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k\right) > 0$ for all $0 \leq k \leq i$, following Lemma is used

Lemma 4

The scalar quadratic forms $J_{f,k}\left(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k\right)$ satisfy the conditions above iff, for all $0 \leq k \leq i$

i) $J_{f,k}\left(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k\right)$ has a minimum with respect to $\left\{x_0, u_0, u_1, \cdots u_k\right\}$.

ii) The $\left\{\check{z}_{k|k}\right\}_{k=0}^{i}$ can be chosen such that the value of $J_{f,k}\left(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k\right)$ at this minimum is positive, viz.

$$\min_{\{x_0, u_0, \cdots, u_k\}} J_{f,k}\left(x_0, u_0, \cdots, u_k, y_0, \cdots, y_k\right) > 0$$

Implications of Lemma 3 and 4

- Lemma 3 provides necessary and sufficient condition for solution to H^∞ problem.
- Lemma 4 guarantees that stationary point of quadratic form is a minimum
- Lemma 4 also guarantees positivity of the quadratic form

To find the stationary point, we make use of Krein-space projection.

- Need a corresponding state-space model

Krein Space State Space Model

The following Krein-space system is introduced:

$$\begin{cases} \boldsymbol{x}_{j+1} = F_j \boldsymbol{x}_j + G_j \boldsymbol{u}_j \\ \begin{bmatrix} \boldsymbol{y}_j \\ \check{z}_{j|j} \end{bmatrix} = \begin{bmatrix} H_j \\ L_j \end{bmatrix} \boldsymbol{x}_j + \boldsymbol{v}_j \end{cases}$$

with

$$\left\langle \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_j \\ \boldsymbol{v}_j \end{bmatrix}, \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_k \\ \boldsymbol{v}_k \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & I\delta_{jk} & 0 \\ 0 & 0 & \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^2 I \end{bmatrix} \delta_{jk} \end{bmatrix}$$

- Corresponding Kalman filter recursion gives aposteriori estimate $\check{z}_{i|i} = \mathcal{F}_f\left(y_0, y_1, \cdots, y_i\right)$ that satisfies $\left\|T_i\left(\mathcal{F}_f\right)\right\|_\infty < \gamma_f$

Outline

Conclusion

- H^∞ problem leads to indefinite quadratic form
- Indefinite quadratic forms are handled well by Krein-space projection
- Recursive solution by making use of Kalman filter theory on Krein-space state-space model